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Semidefinite Relaxations of Fractional Programs via Novel Convexification Techniques *

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Abstract. In a recent work, we introduced the concept of convex extensions for lower semi-continuous functions and studied their properties. In this work, we present new techniques for constructing convex and concave envelopes of nonlinear functions using the theory of convex extensions. In particular, we develop the convex envelope and concave envelope of $z = x/y$ over a hypercube. We show that the convex envelope is strictly tighter than previously known convex underestimators of x/y. We then propose a new relaxation technique for fractional programs which includes the derived envelopes. The resulting relaxation is shown to be a semidefinite program. Finally, we derive the convex envelope for a class of functions of the type $f(x, y)$ over a hypercube under the assumption that f is concave in x and convex in y.

Key words: Convex Envelopes, Convex Extensions, Fractional Programs, Semidefinite Relaxations, Disjunctive Programming

1. Introduction

This paper presents new techniques for developing convex envelopes of nonlinear functions. These techniques are first presented in the context of fractional programs where a new relaxation is developed for nonlinear programs modeled using fractional terms. In the later part of the paper, we present some generalizations and applications of the proposed approach to other nonlinear functions.

Fractional programs have been the subject of numerous research papers [6, 10, 11, 15, 21, 29, 34, 35] survey articles [24, 25], and books [3, 5, 28]. It was shown in [14] that fractional programs are in general $\mathcal{NP}$-Hard. Applications arise in location [3], economics [5], information theory [17], chemical process industry [33, 34], production efficiency [4], stochastic programming [27], and a host of other areas.

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Global optimization approaches to fractional programming have typically employed the following straightforward approach. Each occurrence of $z = x/y$ is replaced by the bilinear equality $zy = x$ which is subsequently linearized using bilinear programming techniques [1, 15, 16]. Queseda and Grossmann [21] developed nonlinear inequalities which, if included in the fractional program, improve the tightness of such a relaxation. Their research revealed that there is a considerable scope for improvement in the current relaxation techniques for x/y. Zamora and Grossmann [34, 35] have since developed the concave envelope characterization of x/y under the assumption that x is nonnegative and y is positive. They also developed certain nonlinear inequalities improving the quality of the convex underestimator of x/y under the same assumption. However, in spite of these improvements, the convex and concave envelope characterization of x/y over an arbitrary rectangular region is still open. In general, very few results are available on the convex and concave envelopes of even simple nonlinear functions (cf. [1, 26, 35]).

In Section 3, we develop the convex and concave envelope of x/y over a rectangular region. We also provide a constructive proof for the nonlinear underestimating inequality developed in [34]. We develop the concave envelope of x/y over the positive orthant using a completely different proof than that in [35]. Our approach in this paper is based on the work of Tawarmalani et al. [31] on convex extensions of lower semi-continuous functions where the authors laid down a theoretical framework of convex extensions and explored their relations to convex envelopes. We present a constructive argument using convex disjunctive programming techniques within the framework of [31] to produce the convex and concave envelopes of x/y over a rectangular set.

The resulting relaxation – including the envelopes of x/y – is shown to be a semidefinite program. This addresses the issue surrounding the solvability of the proposed relaxation, since semidefinite programs – for which an estimate of the solution size is available *a priori* – are polynomially solvable using interior point techniques developed in [2, 19] and subsequent related works. Also, coupled with the works of [7, 13, 20], semidefinite relaxations can now be developed for mixed integer programs with linear, polynomial, and fractional terms. A global optimization algorithm using this relaxation scheme may then be easily constructed along the lines of [30, 35] and shall not be detailed in this work.

The proposed techniques are not limited to the development of the convex and concave envelopes of x/y. In Section 4, we demonstrate that they can be used in a variety of different situations by developing convex/concave envelopes for functions of the type

$$\sum_{i=1}^{n} f_i(x, y_i),$$

where $y_i \in \mathbb{R}^{n_i}$, $x \in \mathbb{R}$, and each f_i is convex in y_i and concave in x.

2. Overview of Convex Extensions

In this section, we summarize the main results of [31]. Convex Extensions were defined as:

DEFINITION 2.1 ([31]). *Let C be a convex set and $X \subseteq C$. A convex extension of a lower semicontinuous function $\phi : X \mapsto \bar{\mathbb{R}}$ over C is any convex function $\eta : C \mapsto \bar{\mathbb{R}}$ such that $\eta(x) = \phi(x)$ for all $x \in X$.*

In the above, $\bar{\mathbb{R}}$ denotes the extended real line. The following result characterizes the constructibility of convex extensions.

THEOREM 2.2 ([31]). *Let C be a convex set and consider an arbitrary collection of faces F_I of C. Then, a convex extension of $\phi : C \mapsto \Re$ restricted to $\cup_{X \in F_I} X$ can be constructed over C if and only if ϕ is convex over each face $X \in F_I$. Further, the convex envelope of ϕ over C is one such convex extension.*

The generating set $G_C(f)$ of a function f over a set C is defined as the set of extreme points of the epigraph of the convexified function projected on the space of the independent variables. The following result characterizes the generating set:

THEOREM 2.3 ([31]). *Let $\phi(x)$ be a lower semi-continuous function on a compact convex set C. Consider a point $x_0 \in C$. Then, $x_0 \notin G_C(\phi)$ if and only if there exists a convex subset X of C such that $x_0 \in X$ and $x_0 \notin G_X(\phi)$. In particular, if for an $\epsilon -$* neighbourhood *$N_\epsilon \subset C$ of x_0, it can be shown that $x_0 \notin G_{N_\epsilon}(\phi)$, then $x_0 \notin G_C(\phi)$.*

Another useful result concerns with the equivalence of convex envelopes of multilinear functions with that of nonlinear functions restricted to the extreme points of a hypercube. In the following, we denote the hypercube in n dimensions as H^n and its extreme points as E^n.

THEOREM 2.4 ([31]). *Consider a nonlinear function $\phi(x) : E^n \mapsto \mathbb{R}$. Let $f(x)$ be the tightest convex extension of ϕ restricted to E^n over H^n. Consider any multilinear function ϕ' such that $\phi'(x) = \phi(x)$ for all $x \in E^n$. There exists at least one such multilinear function. Further, $f(x)$ is the polyhedral convex envelope of $\phi'(x)$ over H^n. If $\phi(x)$ is a multilinear function, then $f(x)$ is its convex envelope.*

In this work, we advocate to use the above results in the construction of relaxations of mathematical programs. The basic technique involves characterizing the generating set of the convex envelope of the functional form, often using Theorem 2.3. The convex envelope is then viewed as the tightest convex extension of the function restricted to the generating set (Theorem 2.2). The convex envelope can then be derived through the use of convexification techniques as long as the generating set can be expressed as a union of a finite number of convex sets. While developing the convex envelope over a hypercube and if the generating set turns

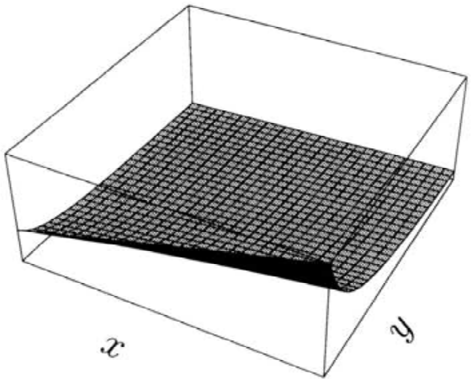

Figure 1. The fractional term x/y in the positive quadrant.

out to be the extreme points, Theorem 2.4 turns out to be an extremely powerful tool in the construction of the envelope.

3. Fractional Programs

In this section, we develop the convex and concave envelopes of x/y over a rectangular feasible region. In Section 3.1 and Section 3.2, we develop the concave and convex envelope of x/y, respectively, over a rectangular region in the positive orthant. In Section 3.3, we establish that the convex envelope is strictly tighter than any previously known convex underestimating inequality for x/y over the positive orthant. In Section 3.4, we relax the restriction that the feasible set be a subset of the positive orthant. In Section 3.5, we demonstrate that the concave and convex envelopes of x/y can be used in a semidefinite programming relaxation for fractional programs.

3.1. CONCAVE ENVELOPE OF x/y

Consider the fractional function x/y over a rectangular subset, $[x^L, x^U] \times [y^L, y^U]$, of the positive quadrant as depicted in Figure 1. Some characteristics of the function are:

- at a fixed value of y, the function is linear;
- at a fixed value of x, the function is convex.

An application of Theorem 2.3 shows that the generating set of the concave envelope of x/y consists of the four corners of the hypercube:

$$G_{[x^L,x^U]\times[y^L,y^U]}\big(\text{conc}\,(x/y)\big) = \{x^L, y^L\} \cup \{x^L, y^U\} \cup \{x^U, y^L\} \cup \{x^U, y^U\}.$$

This follows from the fact that any point, apart from the corner points, can be expressed as a convex combination of neighboring points along either the x axis or the y axis direction. Since the function is convex in both directions, the point in consideration can be eliminated from further consideration and the result follows.

We now develop the concave envelope of x/y. Since the generating set consists of a finite number of points, the concave envelope is polyhedral. A direct application of Theorem 2.4 establishes that the bilinear function which fits the fractional function values at the corner points of the rectangle has the same concave envelope as x/y. Such a bilinear function can be constructed rather easily as:

$$\frac{1}{(x^U - x^L)(y^U - y^L)} \left(\frac{x^L}{y^L}(x^U - x)(y^U - y) + \frac{x^L}{y^U}(x^U - x)(y - y^L) + \frac{x^U}{y^L}(x - x^L)(y^U - y) + \frac{x^U}{y^U}(x - x^L)(y - y^L) \right)$$

and be simplified to:

$$\frac{-xy + xy^L + xy^U}{y^L y^U}.$$

This function is depicted in Figure 2. Now, the development of the concave envelope is trivial using the McCormick envelopes [1]. Algebraically, the concave envelope of x/y over the rectangle $[x^L, x^U] \times [y^L, y^U]$ is given by:

$$\text{concave}\left(\frac{x}{y}\right) = \frac{1}{y^L y^U} \min\left\{ y^U x - x^L y + x^L y^L,\ y^L x - x^U y + x^U y^U \right\}. \tag{1}$$

The concave envelope of x/y was shown to be given by (1) in the work of Zamora and Grossmann [35]. The authors derived the linear inequalities in (1) by using the following relations:

$$\left(\frac{x_i}{x_j} - \frac{x_i^L}{x_j^U}\right)\left(\frac{x_j}{x_j^L} - 1\right) \geqslant 0$$

$$\left(\frac{x_i^U}{y_j^L} - \frac{x_i}{x_j}\right)\left(1 - \frac{x_j}{x_j^U}\right) \geqslant 0,$$

and then verifying that the above formed the concave envelope of x/y.

3.2. CONVEX ENVELOPE OF x/y

The function x/y, as noted earlier, is concave in x for a fixed value of y. An application of Theorem 2.3 to the convex envelope of x/y reveals that the generating set, in this case, is a subset of the faces $x = x^L$ and $x = x^U$. From Theorem 2.2, it follows however that the convex envelope coincides with the function along these

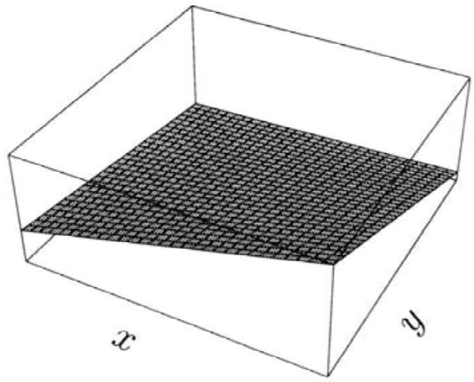

Figure 2. $\frac{-xy+xy^L+xy^U}{y^L y^U}$.

two faces. Since x^L/y and x^U/y are both strictly convex functions, it follows that all the points along the two aforementioned faces belong to the generating set of the convex envelope:

$$G_{[x^L,x^U]\times[y^L,y^U]}\big(\text{conv }(x/y)\big) = \{x^L, y\} \cup \{x^U, y\}.$$

We now characterize the convex envelope of x/y. By applying convex disjunctive programming techniques [23], the epigraph of the convex envelope of x/y may be stated as

$$\left.\begin{aligned} & z \geqslant \tfrac{x^L}{y_a}(1-\lambda) + \frac{x^U}{y_b}\lambda \\ & y^L \leqslant y_a \leqslant y^U \\ & y^L \leqslant y_b \leqslant y^U \\ & y = (1-\lambda)y_a + \lambda y_b \\ & x = x_L + (x^U - x^L)\lambda \\ & 0 \leqslant \lambda \leqslant 1. \end{aligned}\right\} \tag{2}$$

Introduce $y_p = y_a(1-\lambda)$. Then $\lambda y_b = (y - y_p)$. After algebraic manipulations (assuming $0 < \lambda < 1$), (2) above may be restated as:

$$\left.\begin{aligned} & z \geqslant \frac{x^L}{y_p}\left(\frac{x^U - x}{x^U - x^L}\right)^2 + \frac{x^U}{y - y_p}\left(\frac{x - x^L}{x^U - x^L}\right)^2 \\ & y^L(x^U - x) \leqslant y_p(x^U - x^L) \leqslant y^U(x^U - x) \\ & y^L(x - x^L) \leqslant (y - y_p)(x^U - x^L) \leqslant y^U(x - x^L) \end{aligned}\right\} \tag{3}$$

Note that (3) is valid only when $x^L < x < x^U$, since otherwise a division by zero occurs. As $x \to x^L$, (3) converges to the epigraph of x^L/y and, when $x \to x^U$, (3) converges to the epigraph of x^U/y. However, as shall be clear later, the semidefinite programming reformulation of (3) does not result in this instability. This issue is resolved by the introduction of the following variable:

$$z_p \geqslant \frac{x^L}{y_p}\left(\frac{x^U - x}{x^U - x^L}\right)^2.$$

We now get the following reformulation of (3):

$$\text{(R)} \qquad y_p \geqslant \frac{x^L}{z_p}\left(\frac{x^U - x}{x^U - x^L}\right)^2$$

$$y_p \leqslant y - \frac{x^U}{z - z_p}\left(\frac{x - x^L}{x^U - x^L}\right)^2$$

$$y_p \geqslant \max\left\{y^L \frac{x^U - x}{x^U - x^L}, y - y^U \frac{x - x^L}{x^U - x^L}\right\}$$

$$y_p \leqslant \min\left\{y^U \frac{x^U - x}{x^U - x^L}, y - y^L \frac{x - x^L}{x^U - x^L}\right\}$$

$$z - z_p, z_p \geqslant 0.$$

Applying Fourier-Motzkin Elimination to the above constraint set, we can eliminate y_p from the formulation. Note that (R) is valid even along the faces $x = x^L$ and $x = x^U$. The inequality of the form $zy \geqslant x^2$ may be rewritten as $\sqrt{zy} \geqslant x$ to make its convexity apparent.

3.3. RELATION TO EARLIER WORKS

It is obvious that reformulation of the fractional term using (3) produces a relaxation that is tighter than that derived using any other convex underestimator of x/y. In addition, we show that the relaxation of x/y obtained via (3) is at times strictly tighter than the nonlinear inequality developed in [34, 35]:

$$z \geqslant \frac{1}{y}\left(\frac{x + \sqrt{x^L x^U}}{\sqrt{x^L} + \sqrt{x^U}}\right)^2. \tag{4}$$

EXAMPLE 3.1. *We construct a rather simple example where $y^L = y^U$. The feasible region is just the linear segment $[x^L, x^U]$. In this case, the function x/y is* *linear and the equations (3) describe the function exactly as follows:*

$$z \geqslant \frac{x^L}{y^L}\left(\frac{x^U - x}{x^U - x^L}\right) + \frac{x^U}{y^L}\left(\frac{x - x^L}{x^U - x^L}\right) = \frac{x}{y^L}. \tag{5}$$

Since (4) is strictly convex and matches (5) at the end-points, $x = x^L$ and $x = x^U$, (5) defines a chord of (4). Therefore, (5) lies strictly above (4) at all points except the end-points of the interval.

Another set of nonlinear inequalities was developed by [21] for underestimating the fractional term:

$$z \geqslant \frac{x}{y^L} + x^L\left(\frac{1}{y} - \frac{1}{y^L}\right) \tag{6}$$

$$z \geqslant \frac{x}{y^U} + x^U\left(\frac{1}{y} - \frac{1}{y^U}\right) \tag{7}$$

EXAMPLE 3.2. *Let $x^L = y^L = 1$, $x^U = y^U = 3$, and $x = y = 2$. Then, using (6) and (7) we get:*

$$z \geqslant \max\{0.5, 0.5\} = 0.5.$$

However, from (3), we get

$$z \geqslant \frac{1}{4(\sqrt{3} - 1)} + \frac{3}{4(1 - \sqrt{3})} \approx 0.93301.$$

There is something interesting that happens with the above inequalities. (6) and (7) are exact in Example 3.1 and (4) is equal to the convex envelope in Example 3.2. We shall explain why this happens in the sequel. First, we give an example where all the above inequalities are worse than the convex envelope.

EXAMPLE 3.3. *Consider $x^L = 2$, $x^U = 4$, $y^L = 3$, and $y^U = 3.4$. At $x = 3$ and $y = 3.3$, the convex envelope is found by solving:*

$$z \geqslant \frac{1}{2y} + \frac{1}{3.3 - y}$$
$$1.6 \leqslant y \leqslant 1.7.$$

The solution occurs at $y = 1.6$ and generates $z \geqslant 0.900735$. However, using the three inequalities above, we find that: (4) reduces to $z \geqslant 0.883095$, (6) reduces to $z \geqslant 0.878788$, and (7) reduces to $z \geqslant 0.900178$. We have thus provided an example where each of the inequalities (4), (6), and (7) lies strictly below the convex envelope of x/y.

The formal description of the convex envelope when $x^L < x < x^U$ is:

$$(C) \qquad \min \quad \frac{x^L}{y_p}\left(\frac{x^U - x}{x^U - x^L}\right)^2 + \frac{x^U}{y - y_p}\left(\frac{x - x^L}{x^U - x^L}\right)^2 \tag{8}$$

$$y^L(x^U - x) \leqslant y_p(x^U - x^L) \leqslant y^U(x^U - x) \tag{9}$$

$$y^L(x - x^L) \leqslant (y - y_p)(x^U - x^L) \leqslant y^U(x - x^L). \tag{10}$$

This is a single variable optimization problem in y_p for a given point (x, y). It is clear that one potential candidate for y_p is determined by setting the derivative on the objective function to zero. Then, solving the quadratic equation:

$$y_p^2 x^U (x - x^L)^2 - (y - y_p)^2 x^L (x^U - x)^2 = 0,$$

we get the following two solutions:

$$y_p = \frac{\sqrt{x^L}(x^U - x)y}{(\sqrt{x^L} + \sqrt{x^U})(\sqrt{x^L}\sqrt{x^U} - x)} \tag{11}$$

$$y_p = \frac{\sqrt{x^L}(x - x^U)y}{(\sqrt{x^L} - \sqrt{x^U})(\sqrt{x^L}\sqrt{x^U} + x)}. \tag{12}$$

Substituting back in equation (8), we get:

$$z = \frac{1}{y}\left(\frac{x - \sqrt{x^L}\sqrt{x^U}}{\sqrt{x^L} - \sqrt{x^U}}\right)^2 \tag{13}$$

$$z = \frac{1}{y}\left(\frac{x + \sqrt{x^L}\sqrt{x^U}}{\sqrt{x^L} + \sqrt{x^U}}\right)^2. \tag{14}$$

Relaxing $=$ to $\geqslant$ in (13) and (14) produces valid underestimators for x/y. This can be seen by subtracting x/y from both sides of (13) and (14):

$$z - \frac{x}{y} \geqslant \frac{(x - x^L)(x^U - x)}{y(\sqrt{x^L} - \sqrt{x^U})^2} \geqslant 0 \tag{15}$$

$$z - \frac{x}{y} \geqslant \frac{(x - x^L)(x^U - x)}{y(\sqrt{x^L} + \sqrt{x^U})^2} \geqslant 0. \tag{16}$$

Since

$$\frac{(x - x^L)(x^U - x)}{y(\sqrt{x^L} - \sqrt{x^U})^2} \geqslant \frac{(x - x^L)(x^U - x)}{y(\sqrt{x^L} + \sqrt{x^U})^2},$$

(13) is dominated by (14). A more careful analysis of the KKT conditions also shows that (11) is never feasible and hence (13) can be dropped. We carry out this analysis. In deriving (13), we relaxed the bounds on y_p. However, (13) is to be included only if (11) is within bounds specified by (9) and (10). By assumption, $y^L > 0$. (9) and (10) can thus be relaxed to $0 < y < y_p$. It is easy to show that when $x^L < x < x^U$, $x - \sqrt{x^L x^U} \leqslant 0$ implies $y > y_p$, and $x - \sqrt{x^L x^U} \geqslant 0$ implies $y > 0$. At $x = x^L$ or $x = x^U$, (11) and (12) correspond to the same point. Therefore, (11) can be ignored.

We apply a similar analysis to (12). The result from this analysis is however quite different. We show that (14) is the convex envelope of x/y over the positive orthant when no additional bounds on y are known. (14) is valid only if (12) is within bounds specified by (9) and (10) and is the minimizer in (C). (12) lies in the interval $(0, y_p)$ for $x^L < x < x^U$. If (12) is feasible to (9) and (10), then it is the minimizer of (C), since it is the only KKT point in a relaxation of (9) and (10) to the open interval $(0, y_p)$. In particular, if no bounds are available on y except that $y > 0$, then (14) forms the convex envelope of x/y since it is exact when $x = x^L$ or $x = x^U$. In other cases, (14) produces the following underestimator:

$$z \geqslant \frac{1}{y}\left(\frac{x + \sqrt{x^L}\sqrt{x^U}}{\sqrt{x^L} + \sqrt{x^U}}\right)^2. \tag{17}$$

Note that (17) is exactly the nonlinear equation derived in [34, 35]. In [34, 35], however, Equation (17) was neither recognized as one of the participating equations in the convex envelope of x/y nor as the convex envelope of x/y in the absence of bounds on y. We have shown:

THEOREM 3.4. *The convex envelope of x/y over $[x^L, x^U] \times (0, \infty)$ is given by:*

$$z \geqslant \frac{1}{y}\left(\frac{x + \sqrt{x^L}\sqrt{x^U}}{\sqrt{x^L} + \sqrt{x^U}}\right)^2.$$

The only remaining possibilities are that y_p is at one of its bounds in Equations (9) and (10). Using these bounds, we get the following inequality:

$$z \geqslant \min\left\{ \frac{xyx^L - x^2y^L + x^Lx^U(y^L - y)}{y^L(x^Ly - x^Uy + xy^L + x^Uy^L)}, \right.$$
$$\frac{xyx^L - x^2y^U + x^Lx^U(y^U - y)}{y^U(x^Ly - x^Uy + xy^U + x^Uy^U)},$$
$$\frac{xyx^U - x^2y^L + x^Lx^U(y^L - y)}{y^L(-x^Ly + x^Uy + xy^L + x^Ly^L)}, \tag{18}$$
$$\left. \frac{xyx^U - x^2y^U + x^Lx^U(y^U - y)}{y^L(-x^Ly + x^Uy + xy^U + x^Ly^U)} \right\}.$$

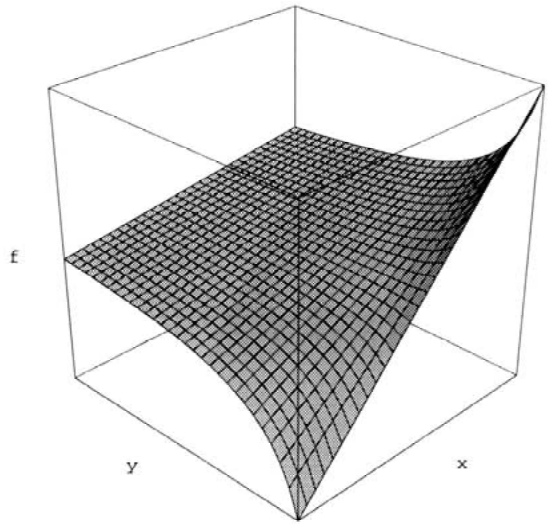

Figure 3. x/y when $0 \in [x^L, x^U]$.

Note that, if $x = x^L(1-\lambda) + x^U(\lambda)$ and y is greater (less) than $y^L(1-\lambda) + y^U(\lambda)$, the first (fourth) term above should be ignored, since it was derived using a value of y_p outside the bound constraints (9)$\big((10)\big)$. Similarly, if y is greater (less) than $y^L\lambda + y^U(1-\lambda)$, the third (second) term should be ignored. A disjunction of this sort cannot be introduced in a mathematical program and hence we will have to resort to the formulation (R) for purposes of constructing the convex relaxation. Nevertheless, if we use a convex programming solver, like MINOS [18] and SNOPT [8], which uses black-box optimization, in the sense that we are required to provide just the function and gradient values at a prespecified point, then the above description of the convex envelope is sufficient.

EXAMPLE 3.5. *Returning to Example 3.3 with* $x^L = 2$*,* $x^U = 4$*,* $y^L = 3$*,* $y^U = 3.4$*,* $x = 3$*, and* $y = 3.3$*, only the second and fourth terms in (18) need to be considered, Hence,* $z \geqslant \min\{0.919118, 0.900735\} = 0.900735$ *resulting in the same bound as derived using the direct computation with (3).*

Since (17) and (18) include all possible solutions of y_p in (C), it is clear that (17) along with (18) with appropriately chosen terms form the convex envelope of the fractional function.

3.4. RELAXING THE POSITIVITY REQUIREMENT

In this subsection, we relax the assumption that x and y belong to the positive orthant and develop the convex and concave envelopes of x/y as long as $0 \notin [y^L, y^U]$ (see Figure 3). To accomplish this, we must derive the convex and concave envelopes of x/y when $0 \in [x^L, x^U]$ and $y^L > 0$. The development of any one of the convex or concave envelope is adequate, since the other is developed in an identical fashion by substituting $u = -x$. Therefore, without loss of generality, we restrict our attention to the convex envelope characterization.

The function x/y is linear in x for a fixed value of y. Therefore, the generating set of the convex envelope is a subset of the faces $x = x^L$ and $x = x^U$. Further, x/y is concave when $x = x^L$. Therefore, the generating set can be written as the following union:

$$(x^L, y^L) \cup (x^L, y^U) \cup \left\{(x^U, y) \mid y^L \leqslant y \leqslant y^U\right\}.$$

Note that the generating set is written as a union of three convex sets. We could convexify the function either over all the three sets together or sequentially in two steps. In the present case, we prefer to do this sequentially. We first convexify x/y over $x = x^L$ and use the convexified function to develop the convex envelope of x/y over $[x^L, x^U] \times [y^L, y^U]$. Since the function is concave over $x = x^L$, its convex envelope can be developed easily as:

$$\frac{x^L(y^L + y^U - y)}{y^L y^U}.$$

Now the convex envelope of x/y is given by:

$$\begin{aligned}
&z \geqslant \frac{x^L(y^L + y^U - y_a)}{y^L y^U}(1 - \lambda) + \frac{x^U}{y_b}\lambda \\
&y^L \leqslant y_a \leqslant y^U \\
&y^L \leqslant y_b \leqslant y^U \\
&y = (1 - \lambda)y_a + \lambda y_b \\
&x = x_L + (x^U - x^L)\lambda \\
&0 \leqslant \lambda \leqslant 1.
\end{aligned}$$

Substituting $y_p = y_a(1 - \lambda)$, $y - y_p = \lambda y_b$, and $\lambda = (x - x^L)/(x^U - x^L)$ we get:

$$\left.\begin{aligned}
&z_p \geqslant \frac{x^L\big(x^L y_p - x(y^L + y^U) + x^U(y^L - y_p + y^U)\big)}{(x^U - x^L)y^L y^U} \\
&(z - z_p)(y - y_p)(x^U - x^L)^2 \geqslant x^U(x - x^L)^2 \\
&y^L(x^U - x) \leqslant y_p(x^U - x^L) \leqslant y^U(x^U - x) \\
&y^L(x - x^L) \leqslant (y - y_p)(x^U - x^L) \leqslant y^U(x - x^L) \\
&z - z_1, z_1 \geqslant 0.
\end{aligned}\right\} \quad (19)$$

We have developed the convex and concave envelopes of x/y as long as $y \neq 0$. This assumption is not restrictive. As y approaches zero from above (below) the function takes arbitrarily large (small) values forcing the concave (convex) envelope to infinity (-infinity) and the function is not well defined at $y = 0$.

3.5. SEMIDEFINITE RELAXATION OF x/y

We now show that nonlinear convex constraints in (R) can be represented as linear matrix inequalities using the *Schur complement* [32]. We denote the matrix inequality expressing positive semi-definiteness of A, as $A \succeq 0$. Consider the following matrix inequality:

$$A = \begin{pmatrix} y_p(x^U - x^L)^2 & \sqrt{x^L}(x^U - x) \\ \sqrt{x^L}(x^U - x) & z_p \end{pmatrix} \succeq 0. \tag{20}$$

This inequality expresses the equation

$$z_p y_p (x^U - x^L)^2 - x^L (x^U - x)^2 \geqslant 0,$$

since $y_p(x^U - x^L)^2 \geqslant 0$ and $z_p \geqslant 0$. Similarly,

$$B = \begin{pmatrix} (y - y_p)(x^U - x^L)^2 & \sqrt{x^U}(x - x^L) \\ \sqrt{x^U}(x - x^L) & z - z_p \end{pmatrix} \tag{21}$$

The above inequality expresses the equation

$$(z - z_p)(y - y_p)(x^U - x^L)^2 - x^U (x - x^L)^2 \geqslant 0,$$

since $(y - y_p)(x^U - x^L)^2 \geqslant 0$ and $z \geqslant z_p$. The following expresses the lower bound on y_p:

$$C = \begin{pmatrix} y_p - y^L \dfrac{x^U - x}{x^U - x^L} & \\ & y_p - y + y^U \dfrac{x - x^L}{x^U - x^L} \end{pmatrix} \succeq 0. \tag{22}$$

The equation below expresses the upper bound on y_p:

$$D = \begin{pmatrix} y^U \dfrac{x^U - x}{x^U - x^L} - y_p & \\ & y - y_p - y^L \dfrac{x - x^L}{x^U - x^L} \end{pmatrix} \succeq 0. \tag{23}$$

Cumulatively, the SDP relaxation of x/y may be expressed as:

$$\begin{pmatrix} A & & & \\ & B & & \\ & & C & \\ & & & D \end{pmatrix} \succeq 0. \tag{24}$$

The semidefinite relaxation of x/y described above is second-order cone representable. Using the equivalence [12]:

$$yz \geqslant x^2,\ y \geqslant 0, z \geqslant 0 \Longleftrightarrow \left\| \begin{pmatrix} 2x \\ y - z \end{pmatrix} \right\| \leqslant y + z \tag{25}$$

the second-order cone representation can be obtained as:

$$\begin{aligned}
&\left\| \begin{pmatrix} 2(1-\lambda)\sqrt{x^L} \\ z_p - y_p \end{pmatrix} \right\| \leqslant z_p + y_p \\
&\left\| \begin{pmatrix} 2\lambda\sqrt{x^U} \\ z - z_p - y + y_p \end{pmatrix} \right\| \leqslant z - z_p + y - y_p \\
&y_p \geqslant y^L(1-\lambda),\ y_p \geqslant y - y^U\lambda \\
&y_p \leqslant y^U(1-\lambda),\ y_p \leqslant y - y^L\lambda \\
&x = x_L + (x^U - x^L)\lambda \\
&z_p, u, v \geqslant 0, z_c - z_p \geqslant 0 \\
&0 \leqslant \lambda \leqslant 1.
\end{aligned} \tag{26}$$

Using an almost identical procedure, it is easy to show that (19) can also be transformed into a semidefinite program using *Schur Complements* or into second-order cone program using (25). We have thus shown that fractional programs can be relaxed using semidefinite relaxations. Coupled with similar results for indefinite quadratic programs [7, 9], our result provides a systematic means for constructing semidefinite relaxations for general factorable programs.

3.6. ENVELOPES OF $(ax + by)/(cx + dy)$

Consider the function $f(x, y) = (ax + by)/(cx + dy)$ over a rectangle $[x^L, x^U] \times [y^L, y^U]$ in the positive orthant. We assume that a, b, c, and d are non-negative constants and at least one of c and y is strictly positive. In this sense, the function $f(x, y)$ is a slight generalization of x/y as seen by setting $b = c = 0$ and $a = d = 1$. We develop the convex envelope of $f(x, y)$ in this section. The concave envelope can be easily developed by a similar treatment.

The case $ad = bc$ is trivial since the function is either a constant or undefined. Without loss of generality, we assume $ad > bc$. Note that $f(x, y^0)$ is concave for a fixed $y^0 > 0$. When $c = 0$, the result is obvious. Otherwise, the concavity of $f(x, y^0)$ follows from the following identity:

$$\frac{ax + by^0}{cx + dy^0} = \frac{a}{c} - \frac{1}{c}\left(\frac{(ad - bc)y^0}{cx + dy^0}\right).$$

For any fixed x^0, $f(x^0, y)$ is convex since:

$$\frac{ax^0 + by}{cx^0 + dy} = \frac{b}{d} + \frac{1}{d}\left(\frac{(ad - bc)x^0}{cx^0 + dy}\right) \tag{27}$$

and $ad > bc$, $d > 0$ and $cx^0 + dy > 0$. The epigraph of the convex envelope of $f(x, y)$ is thus expressible as the convex hull of $A \cup B$ where

$$A = \left\{(z_a, x^L, y_a) \mid z_a \geqslant f(x^L, y_a), y^L \leqslant y_a \leqslant y^U\right\}$$

and

$$B = \left\{(z_b, x^U, y_b) \mid z_b \geqslant f(x^U, y_b), y^L \leqslant y_b \leqslant y^U\right\}.$$

We denote a point in the epigraph of the convex envelope of $f(x, y)$ by (z, x, y). Introducing $\lambda = (x - x^L)/(x^U - x^L)$, $y_p = y_a(1 - \lambda)$, and $z_p = z_a(1 - \lambda)$, the epigraph of the convex envelope of $f(x, y)$ can be written as:

$$\left.\begin{aligned}
&dz_p(cx^L(1-\lambda) + dy_p) = (ad - bc)x^L(1-\lambda)^2\\
&(dz - b - dz_p)(cx^U\lambda + dy - dy_p) = (ad - bc)x^U\lambda^2\\
&y_p \geqslant \max\left\{y^L(1-\lambda), y - y^U\lambda\right\}\\
&y_p \leqslant \min\left\{y^U(1-\lambda), y - y^L\lambda\right\}\\
&x = x_L + (x^U - x^L)\lambda\\
&z_p, u, v \geqslant 0, z - z_p \geqslant b/d\\
&0 \leqslant \lambda \leqslant 1
\end{aligned}\right\} \tag{28}$$

using a procedure similar to that described in Section 3.2. A second-order cone representation of (28) is easily derived using (25). Analyzing the KKT conditions as in Section 3.3, the following convex underestimating inequality is derived:

$$f(x, y) \geqslant \frac{(ad - bc)(x + \sqrt{x^L}\sqrt{x^U})^2}{d(\sqrt{x^L} + \sqrt{x^U})^2(cx + dy)} + \frac{b}{d} \tag{29}$$

and shown to be the convex envelope of $f(x, y)$ over $[x^L, x^U] \times (0, \infty)$.

4. Generalizations

The techniques presented in this paper are fairly general and find applications in developing convex/concave envelopes in a wide variety of situations. We illustrate this by developing convex/concave envelopes of $f(x, y)$ where f is lower semi-continuous concave in x and convex in y. It may be pointed out that we do not assume that y is a scalar. The generating set of the convex envelope of f over a rectangular region is then the set of faces: $x = x^L$ and $x = x^U$. By disjunctive programming techniques, the convex envelope is given by:

$$\left.\begin{aligned} & z \geqslant f(x^L, y_a)(1-\lambda) + f(x^U, y_b)\lambda \\ & y^L \leqslant y_a \leqslant y^U \\ & y^L \leqslant y_b \leqslant y^U \\ & y = (1-\lambda)y_a + \lambda y_b \\ & x = x_L + (x^U - x^L)\lambda \\ & 0 \leqslant \lambda \leqslant 1. \end{aligned}\right\} \tag{30}$$

In a similar vein to [23], we define a positively homogeneous function g associated with $f(\cdot, y)$ by the following relation:

$$g(\cdot, \lambda, y) = \begin{cases} \lambda f(\cdot, \lambda^{-1}y) & \lambda > 0 \\ 0 & \lambda = 0, \text{ and } y = 0 \\ +\infty & \lambda = 0, \text{ and } y \neq 0. \end{cases}$$

Since the epigraph of g is convex, g is jointly convex in λ and y. Also, it follows from Theorem 8.2 in [23] that g is closed if f is bounded in the space under consideration. Introduce the variable $y_p = y_a(1-\lambda)$. After algebraic manipulations, we get the following form of (30):

$$\left.\begin{aligned} & z \geqslant g\left(x^L, \frac{x^U - x}{x^U - x^L}, y_p\right) + g\left(x^U, \frac{x - x^L}{x^U - x^L}, y - y_p\right) \\ & y^L(x^U - x) \leqslant y_p(x^U - x^L) \leqslant y^U(x^U - x) \\ & y^L(x - x^L) \leqslant (y - y_p)(x^U - x^L) \leqslant y^U(x - x^L). \end{aligned}\right\} \tag{31}$$

Therefore, whenever there is a way to write the mathematical formulation of g, the convex envelope of $f(x, y)$ can be developed as (31). It is possible to generalize the above to the case when x is a vector. However, the generalization is not only unnecessary but restrictive since the same effect is achieved by convexifying the function sequentially using the x variables one at a time. Generalizations to Cartesian products of polytopes (instead of a hypercube) as the feasible space can be easily accomplished. However, such an application does not serve to clarify the proposed concepts any further than already achieved through the previous example.

4.1. CONVEX ENVELOPE OF $f(x)y^2$

We consider the function $f(x)y^2$ over a rectangular region. We assume that $f(x) \geqslant 0$ over the feasible region. We provide an illustration of $x^{0.8}y^2$ when $0 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$ in Figure 4. We assume that f is a concave function of x. Then, the function is convex in y and concave in x. Using (31), the convex envelope is expressed as:

$$\left.\begin{aligned}
&\min\ g_1 + g_2\\
&g_1(x^U - x) \geqslant f(x^L)y_p^2(x^U - x^L)\\
&g_2(x - x^L) \geqslant f(x^U)(y - y_p)^2(x^U - x^L)\\
&y^L(x^U - x) \leqslant y_p(x^U - x^L) \leqslant y^U(x^U - x)\\
&y^L(x - x^L) \leqslant (y - y_p)(x^U - x^L) \leqslant y^U(x - x^L)\\
&g_1, g_2 \geqslant 0.
\end{aligned}\right\} \tag{32}$$

The various candidates for the solution are found by setting the derivative of g_1+g_2 to zero and setting y_p to one of the bounds. The first candidate is found by solving

$$2f(x^L)y_p\frac{x^U - x^L}{x^U - x} - 2f(x^U)(y - y_p)\frac{x^U - x^L}{x - x^L} = 0.$$

The resulting solution is

$$y_p = \frac{f(x^U)(x - x^U)y}{(x - x^L)f(x^L) + (x - x^U)f(x^U)}.$$

Then:

$$\begin{aligned}
z \geqslant \min\Bigg\{ &\frac{(x^L - x^U)y^2 f(x^L)f(x^U)\big((x^L - x)f(x^L) + (x - x^U)f(x^U)\big)}{\big((x - x^L)f(x^L) + (x - x^U)f(x^U)\big)^2},\\
&\frac{(x - x^U)(y^U)^2 f(x^L)}{(x^L - x^U)} - \frac{(x^L y - xy^U - x^U y + x^U y^U)^2 f(x^U)}{(x - x^L)(x^L - x^U)},\\
&\frac{(x - x^U)(y^L)^2 f(x^L)}{(x^L - x^U)} - \frac{(x^L y - xy^L - x^U y + x^U y^L)^2 f(x^U)}{(x - x^L)(x^L - x^U)},\\
&\frac{(x^U y - xy^L - x^L y + x^L y^L)^2 f(x^L)}{(x - x^U)(x^L - x^U)} + \frac{(x - x^L)(y^L)^2 f(x^U)}{(x^U - x^L)},\\
&\frac{(x^U y - xy^U - x^L y + x^L y^U)^2 f(x^L)}{(x - x^U)(x^L - x^U)} + \frac{(x - x^L)(y^U)^2 f(x^U)}{(x^U - x^L)}\Bigg\}.
\end{aligned} \tag{33}$$

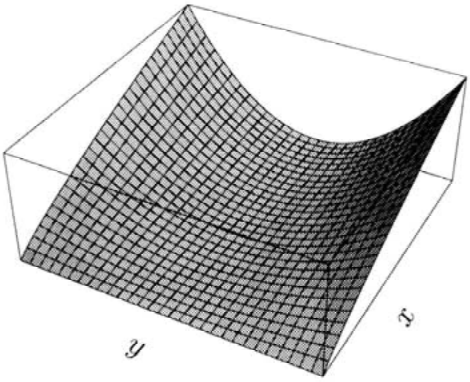

Figure 4. $x^{0.8}y^2$.

Note that, at any given point, out of the five terms only those should be considered which were derived with y_p within the bounds in (32). We now investigate the functions $x^a y^2$, $0 < a < 1$ (see Figure 4) and $\log_{10}(9x+1)y^2$ (see Figure 5) when $x^L = 0$, $x^U = 1$, $y^L = -1$, $y^U = -1$. In this case, (33) reduces to:

$$z = \begin{cases} 0 & x = 0 \\ 0 & y + x \leqslant 1 \text{ and } y \geqslant 0 \\ (x+y-1)^2/x & y + x \geqslant 1 \text{ and } x \neq 0 \\ 0 & y - x \leqslant 1 \text{ and } y < 0 \\ (1-x+y)^2/x & x - y \geqslant 1 \text{ and } x \neq 0. \end{cases}$$

As in Section 3.4, the assumption that $f(x) \geqslant 0$ can be relaxed. All that is needed is that we convexify the function $f(x)y^2$ over $x = x^L$ if $f(x^L) \leqslant 0$ and over $x = x^U$ if $f(x^U) \leqslant 0$. It is easy to see that the relaxation (32) can be transformed to linear matrix inequalities and therefore included in a semidefinite relaxation of $f(x)y^2$.

4.2. CONVEX ENVELOPE OF $f(x)/y$

In this section, we consider a slightly more general form of the fractional function x/y. We assume that $y > 0$ and $f(x)$ is a concave function of x. Even though we could follow the same construction as in Section 4.1, we shall make use of the convex envelope of the fractional function to develop the convex envelope. This is a simple technique which may be used in more general settings and we use it in this context to illustrate its use. Since the generating set of this function is the same as that of x/y, the following function has the same convex envelope as $f(x)/y$:

$$\frac{1}{y}\left(f(x^L)\frac{x^U - x}{x^U - x^L} + f(x^U)\frac{x - x^L}{x^U - x^L}\right),$$

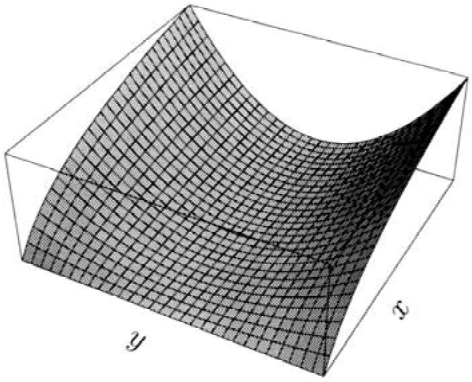

Figure 5. $\log_{10}(9x+1)y^2$.

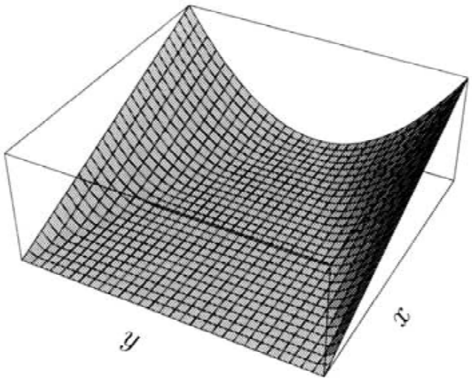

Figure 6. Convex Envelope of $x^{0.8}y^2$.

which in turn can be generated knowing the convex envelope of x/y. This procedure is equivalent to introducing a variable f as $f \geqslant f(x)$ and then relaxing the inequality using the convex envelope of $f(x)$ to get the following:

$$f \geqslant \left(f(x^L)\frac{x^U - x}{x^U - x^L} + f(x^U)\frac{x - x^L}{x^U - x^L} \right).$$

It is clear that the relaxations developed in this case can also be transformed in a semidefinite program.

4.3. SUMMATION OF FUNCTIONS

The following result appears in [22] in a slightly different form:

THEOREM 4.1 ([22]). *Consider a hypercube $P = H^{1+\sum_{i=1}^{n} n_i}$. Let $x \in \mathbb{R}$ and $y_i \in \mathbb{R}^{n_i}$. Assume $f_i(x, y_i)$ is a continuous function for each $i \in \{1, \ldots, n\}$. Assume* *further that each* $f_i(x, y_i)$ *is concave in* x. *Then:*

$$\text{convex}_P\left(\sum_{i=1}^{n} f_i(x, y_i)\right) = \sum_{i=1}^{n} \text{convex}_P\, f_i(x, y_i).$$

In Section 4, we developed the convex envelope of each of the functions $f_i(x, y_i)$, under the additional assumption that $f_i(x, y_i)$ is convex in y_i. Note that if this assumption is not satisfied we could in principle convexify along the function in the y_i space before proceeding. It follows directly from Theorem 4.1 that using the techniques presented in this paper, we can develop the convex envelope of functions of the form:

$$\sum_{i=1}^{n} f_i(x, y_i)$$

given that each $f_i(x, y_i)$ is concave in x and convex in y_i. Quite a few functions fall into this category. As an example, consider

$$f(x)\sum_{i=1}^{n}\sum_{j=-p}^{k} a_{ij} y_i^j$$

where f is a concave function, $a_{ij} > 0$ for $i = 1, \ldots, n$; $j = -p, \ldots, k$ and $y_i > 0$.

5. Conclusions

The purpose of this paper has been to demonstrate that convex extensions can be used to develop convex envelopes and, in general, convex relaxations of nonlinear nonconvex programs. Using this tool, we have developed a semidefinite programming relaxation for fractional programming problems. As a result, new algorithms may be developed for factorable nonlinear programming problems incorporating these relaxations and solving semidefinite programs instead of the traditional linear programming relaxations. Since the use of semidefinite programming relaxations in branch and bound codes is an ongoing research area, we shall not dwell on it more. Further, this relaxation strategy may prove useful in developing approximation algorithms for fractional programs.

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